On rank jumps on families of elliptic curves

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The new results are from 3 distinct collaborations with Dan Loughran (Bath- UK), Renato Dias (UFRJ) and Hector Pasten (PUC-Chile)

Plan

- **Motivation**
- **Definitions and examples**
- **The problem and different methods**
- **More on the geometric method: Results**

Ranks of elliptic curves

Let k be a number field and E/k an elliptic curve.

$E : y^2 = x^3 + ax + b$, with $a, b \in k$.

Mordell-Weil Theorem: $E(k) \simeq \mathbb{Z}^{r(a,b)} \oplus Tors_{a,b}$.

Consider a family of elliptic curves:

$$(\star) \quad E_t : y^2 = x^3 + a(t)x + b(t), \text{ with } a(t), b(t) \in k[t].$$

For $t \in k$ such that $\Delta(t) \neq 0$, we have $E_t(k) \simeq \mathbb{Z}^{r_t} \oplus Tors_t$.

Natural Question: How does r_t behave as t varies?

TODAY: We'll use surfaces to deal with this question.

Elliptic surface

A smooth projective surface S is called an *elliptic surface* if $\exists \pi : S \to B,\ s.t.$

- $\pi^{-1}(t)$ is a smooth curve of genus 1, for almost all $t \in B$
- $\exists \sigma : B \to S$, a section

We suppose moreover that

- there is at least one singular fiber
- π is relatively minimal

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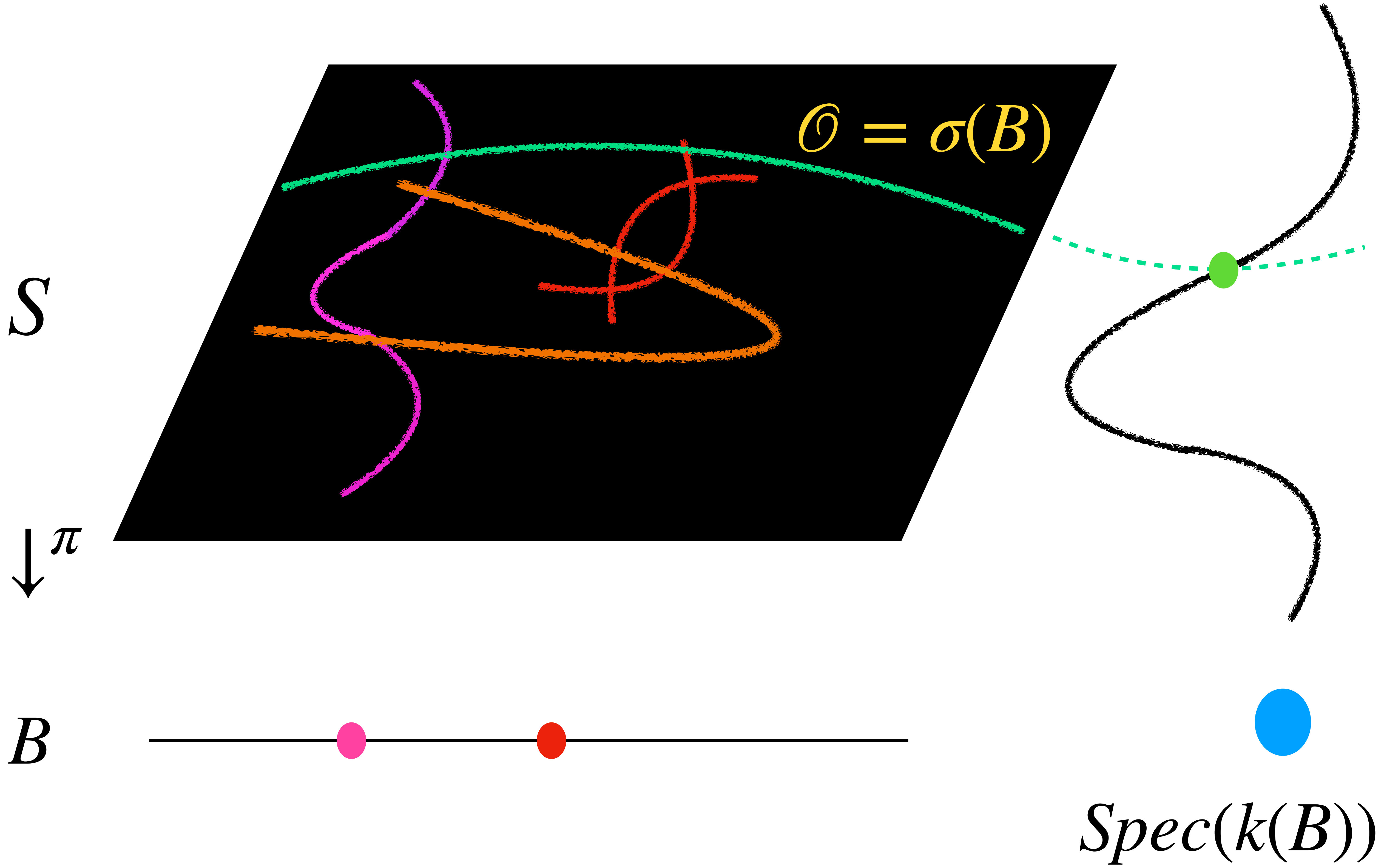


$$Y^2 = X^3 + aX + b\,;\quad a, b \in k(B)$$

In orange: a multisection

Why do we care?

Elliptic surfaces appear in many places

- Shioda-Tate: $\mathrm{NS}(S)/T \simeq \mathrm{MW}(\pi)$
- Zariski density/potential density (Bogomolov-Tschinkel, S.- van Luijk)
- k-unirationality of conic bundles (Kollár-Mella)
- Dense sphere packings (Shioda, Elkies)
- Error correcting codes (S. - Várilly-Alvarado - Voloch)
- High rank elliptic curves over $\mathbb{Q}$ (Elkies - record: 28)

An example

Rational elliptic surfaces

Consider F, G two plane cubics. Then

$F \cap G = 9$ points counted with multiplicities and we have

Blow up $F \cap G$

$$\begin{array}{ccc} & & S \\ & \swarrow & \downarrow \pi \\ \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \end{array}$$

$$(x : y : z) \mapsto (F(x, y, z) : G(x, y, z))$$

Arithmetic of elliptic surfaces

Given a number field k

The Mordell-Weil theorem tells us that:

- For the special fibers $E_t := \pi^{-1}(t)$ with $t \in B(k)$:

$$E_t(k) = \mathbb{Z}^{r_t} \oplus \mathrm{Tors}_t \, .$$

- For the generic fiber:

$$\mathscr{E}_\eta(k(B)) = \mathbb{Z}^r \oplus \mathrm{Tors}.$$

From now on: *rank* = Mordell-Weil rank, and r_t denotes the rank of the special fiber E_t and r denotes the rank of the generic fiber.

Ranks of elliptic curves in families

$(\star)\quad E_t : y^2 = x^3 + a(t)x + b(t)$, with $a(t), b(t) \in k[t]$.

Natural Question: How does r_t behave as t varies?

Given $i \in \mathbb{N}$ and $\mathcal{G}_i := \{t \in \mathbb{P}^1(k); r_t = i\} \subset \mathbb{P}^1(k)$, what can we say about $\#\mathcal{G}_i$?

Néron-Silverman's Specialization Theorem: $r_t \geq r$ for all but finitely many t.

So $i < r \Rightarrow \#\mathcal{G}_i < \infty$.

What about $\mathcal{G}_i$, for $i \geq r$?

We'll look at $\mathcal{F}_{r+i} := \{t \in \mathbb{P}^1(k); r_t \geq r + i\}$.

Ranks of elliptic curves in families

Néron and Silverman Specialization Theorems tell us that:

$r_t \geq r$, for all but finitely many $t \in B(k)$.

More precisely:

Néron: outside a **THIN** set.

Silverman: outside a set of bounded height.

Can we say more?

When $r_t > r$ we say that the **rank jumps.**

TODAY: Does the **rank jump**? **Where and how large is the jump**?

Methods

- Root Numbers
- Height Theory
- Base change

Root numbers

Given an elliptic curve E/k. The root number of E is the sign of the functional equation:

$$\tilde{L}(E, s) = W(E)\, \tilde{L}(E, 2 - s).$$

Parity conjecture: $W(E) = (-1)^{\text{rank}(E)}$.

In other words: *The parity of the rank of an elliptic curve over a number field is determined by its root number.*

Variation of the root number ⇒ Rank jump

Constant root number with different "parity" from the generic rank ⇒ Rank jump.

Variation of root numbers in families

Isotrivial families (Rohrlich, Gouvêa, Mazur, Várilly-Alvarado, Dokchitser^2, Desjardins)

Ex: $Y^2 = X^3 - (1 + T^4)X$, $W(E_t) = -1, \forall t$ and hence $\#\mathcal{F}_{r+1} = \infty$.

Non-isotrivial families

Expected: Both +1 and -1 occur infinitely often.

Holds under major conjecture and known under hypothesis on the degree of the coefficients.

Height theory approach

Definition: $P \in E_t(\mathbb{Q})$ is a **division point** if $\exists n \in \mathbb{N}$ s.t. $n \cdot P \in \mathrm{Sec}(\pi)(\mathbb{Q})$.

Let $U \subset S$ be a Zariski open. We denote by U_{div} the set of division points in U.

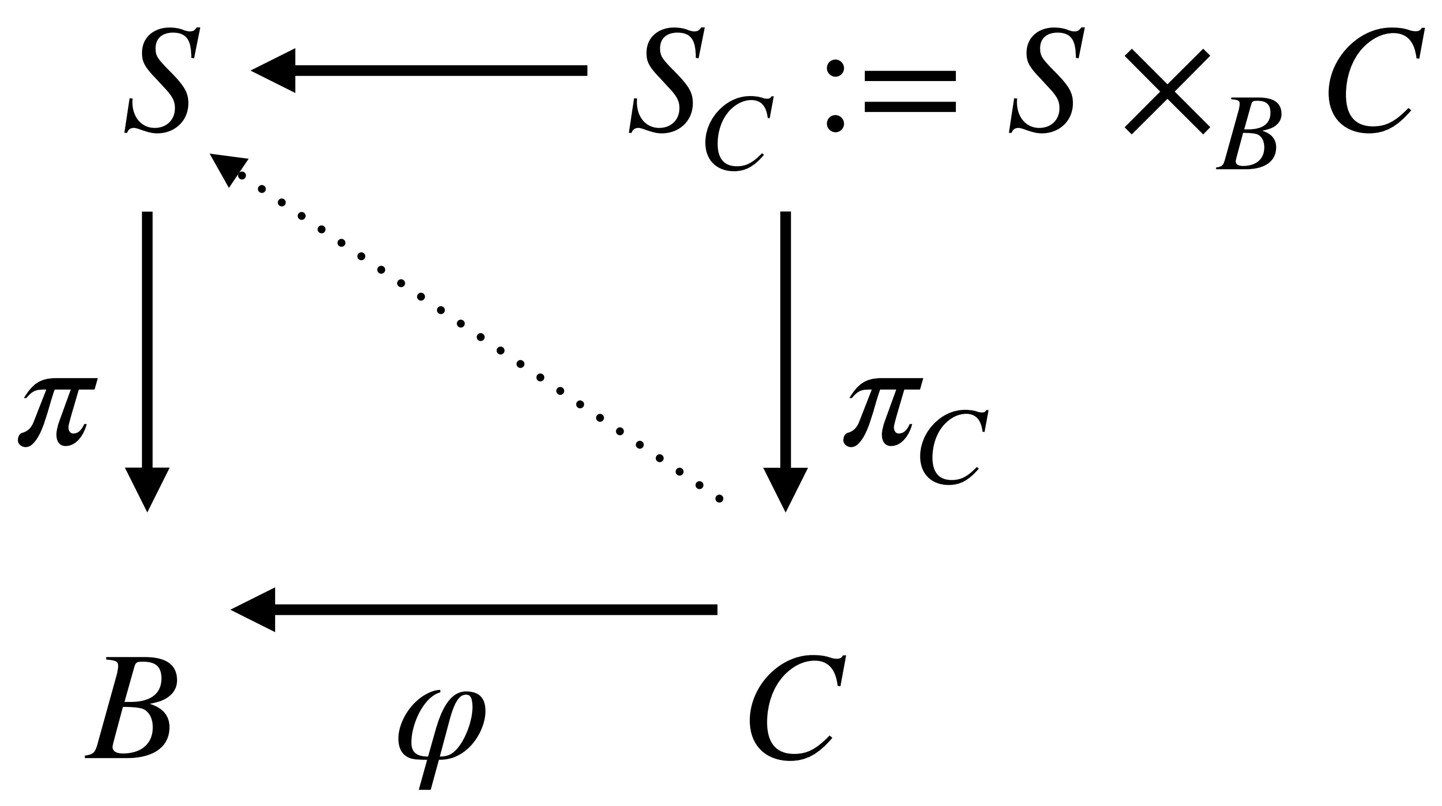
Idea: Count division points of bounded height on fibers and compare with total count (of bounded height).

Billard (2000): Let S be a $\mathbb{Q}-$rational elliptic surface and $D \subset S$ an ample divisor.
There is $\delta > 0$ s.t. $\forall U \subset S$

$N(U(\mathbb{Q}), H_D, B) \gg B^\delta$ and $N(U_{div}(\mathbb{Q}), H_D, B) \ll B^{\delta/2}$

Corollary (Billard): $\#\mathcal{F}_{r+1} = \infty$.

Geometric approach: base change



- π_C is an elliptic fibration
- Sections of π induce sections of π_C
- New section $\sigma_C : C \to S \times_B C$
- Hence $\mathrm{rk}(S_C(k(C))) \geq r = \mathrm{rk}(S(k(B)))$
- If σ_C independent of sections of π then:

For $t \in \varphi(C(k)) \subset B(k)$ we have $r_t \geq r + 1$.

Interesting when $\#C(k) = \infty$ because then we get rank jump on an infinite set!

Rank jumps by base changing

- S. (2011): If S is $k-$unirational then

$$\#\mathscr{F}_{r+1} = \infty\,.$$

- S. (2011) If moreover S has two conic bundle structures then

$$\#\mathscr{F}_{r+2} = \infty\,.$$

What about the quality of these sets?

Quality - Expectation

- Silverman conjectured in the 80's that

$r_t = r$ or $r + 1,$

for $100\,\%$ of the fibers when ordered by height

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Thin Sets

Given an algebraic variety V over k. A subset $T \in V(k)$ is said to be:

- Of ***type 1*** if it is contained in a proper Zariski closed subset.
- Of ***type 2*** if is contained in the image of the $k-$points of a dominant morphism of degree at least 2

$$\phi : W \to V, \text{ so } T \subset \phi(W(k)) \subset V(k).$$

T is called ***THIN*** if it is contained in a finite union of subsets of types 1 and 2.

V is said to satisfy the **HILBERT PROPERTY over k**, if $V(k)$ is not thin.

Examples

A. Over number fields, $\mathbb{P}^n$ satisfies the Hilbert Property, for all n.

B. The set of $\square$'s in a number field is THIN. Indeed, they lie in the image of the degree 2 map $t \mapsto t^2$.

Our contribution on rational elliptic surfaces

Thm. A (Loughran, S. 2019): Let $\pi : S \to \mathbb{P}^1$ be a geometrically rational elliptic surface such that π admits a bisection of arithmetic genus zero then

$$\mathscr{F}_{r+1} = \{t \in \mathbb{P}^1(k); r_t \geq r + 1\} \text{ is not thin.}$$

Thm. B (Loughran, S. 2019): If moreover π has at most one non-reduced fiber OR admits a 2-torsion section defined over k, then

$$\mathscr{F}_{r+2} = \{t \in \mathbb{P}^1(k); r_t \geq r + 2\} \text{ is not thin.}$$

Our contribution on rational elliptic surfaces

Thm. C (Dias, S. 2021): Let $\pi : S \to \mathbb{P}^1$ be a geometrically rational elliptic surface such that one of the following holds:

$i)$ It has a non-reduced fibre of type II^*, III^* or I_n^*, for $2 \leq n \leq 4$;

$ii)$ It has a non-reduced fibre of type IV^*, I_1^* or I_0^* and a reducible reduced singular fibre;

then,

$$\mathscr{F}_{r+3} = \{t \in \mathbb{P}^1(k); r_t \geq r + 3\} \text{ is not thin.}$$

Our contribution on K3 surfaces

Thm. D (Pasten, S. 2022): Let $\pi : S \to \mathbb{P}^1$ be a non-isotrivial elliptic K3 surface without non-reduced fibers. Suppose that there is a different elliptic fibration $\nu : S \to \mathbb{P}^1$ over k. Then the following are equivalent:

$i)$ $X(k)$ is Zariski dense in X;

$ii)$ $\#\mathscr{F}_{\pi,r+1} = \infty$;

$iii)$ $\mathscr{F}_{\pi,r+1}$ is not thin in $\mathbb{P}^1(k)$

Proof of Theorem D: Main tools

I) We consider a base change by a multisection;

II) Check that the multisection induces a $\mathbb{Z}$-linearly independent section;

III) Check that the subset of the base of the fibration where the rank jumps is not thin;

Proof of Theorem D: Main tools

I) We consider a base change by a multisection;

Take a fiber of the other elliptic fibration. ✓

II) Check that the multisection induces a $\mathbb{Z}$-linearly independent section;

Check that it is *Saliently ramified.* ✓

III) Check that the subset of the base of the fibration where the rank jumps is not thin;

Use the fact that we have infinitely many!

How does one show that a subset $\mathscr{B}$ of the line is NOT THIN?

We have to check that given a finite number of arbitrary covers

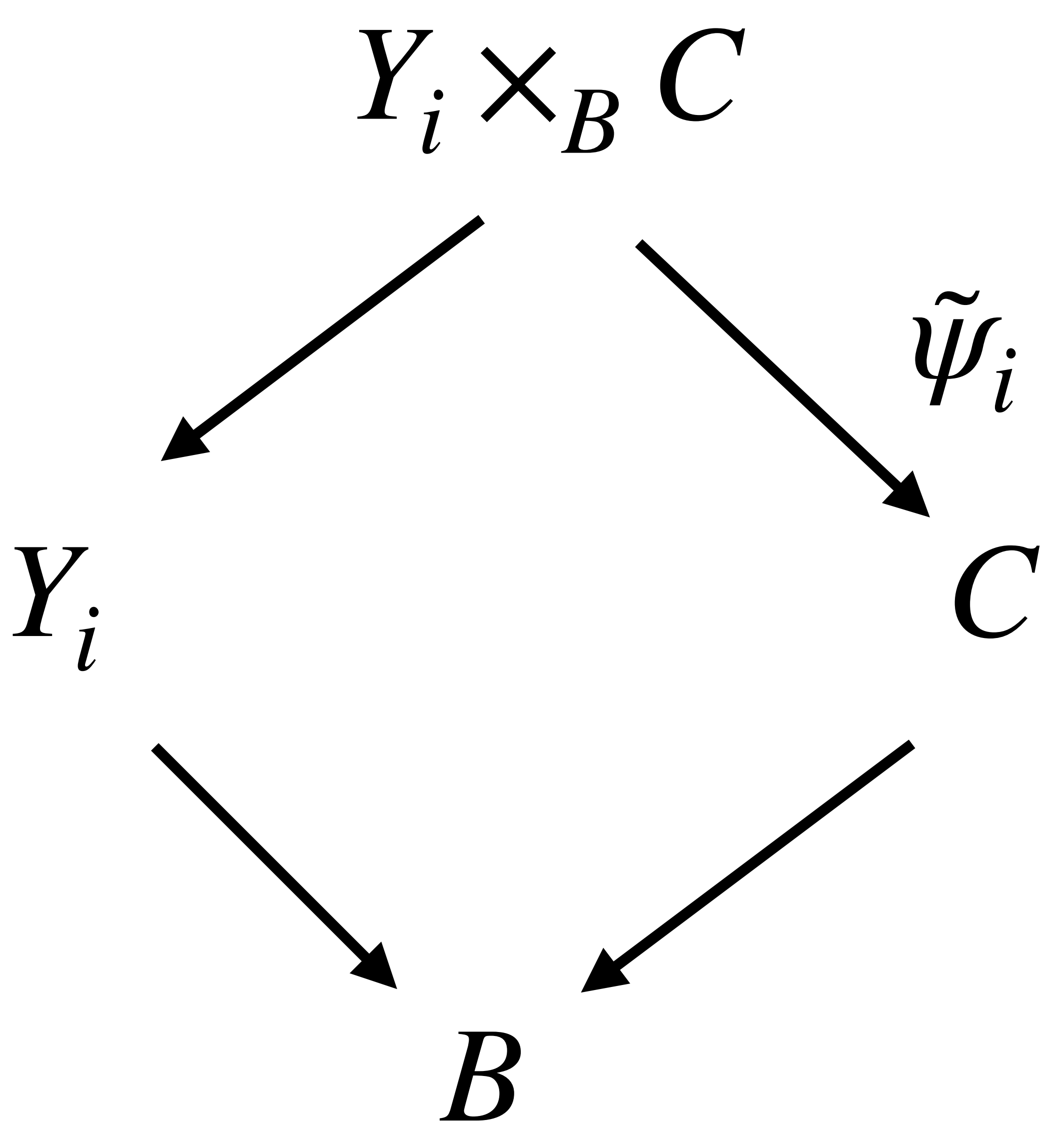
$$\phi_i : Y_i \to \mathbb{P}^1, i = 1,\cdots, n$$

there exists $t \in (\mathbb{P}^1(k) \cap \mathscr{B}) \backslash (\cup_i \phi_i(Y_i(k)))$.

Avoiding the covers

Given a finite number of covers $\psi_i : Y_i \to B$ we have to find $P \in C(k)$ such that

$\varphi(P) \notin \cup\, \psi_i(Y_i(k))$.



If $Y_i \times_B C$ is an integral curve then

Hurwitz's formula tells us that $g(Y_i \times_B C') \geq 2$.

By Faltings' theorem, $(Y_i \times_B C)(k)$ is finite.

Since $\mathrm{rk}(G_s(k)) > 0$ $\exists P \in C(k) \backslash \tilde{\psi}_i((Y_i \times_B C)(k))$, i.e.,

$\varphi(P) \in B \backslash (\cup_i \psi_i(Y_i(k)))$.

$Y_i \times_B C$ is integral. Indeed, we can take T as the set of branch points of ψ_i's.

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Thank you!